A MODEL FOR DESIGNING COGNITION-AND-INSTRUCTION-BASED GOAL TRAJECTORIES FOR RESEARCH IN K–6 MATH CURRICULA

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This paper describes a model for building cognition-and-instruction-based goal trajectories (GT) in the context of a study that examines the validity of curriculum-embedded assessments. The model consists of six design processes and two constraints. The GT is constructed from curriculum-specified learning goals as well as developmental progressions and learning trajectories derived from empirical research. The GT is designed to inform both the selection of assessment activities for data collection and the interpretation of empirical results. Two primary results of the design process are presented: (a) a goal trajectory for promoting algebraic understanding and (b) the relationships between the trajectory and features of the Common Core State Standards. Implications of the design model for building GTs that can be used to assess student reasoning are discussed.

Keywords: Assessment and Evaluation; Curriculum Analysis; Learning Trajectories

Introduction

Learning trajectories are constructs designed to approximate variability and change in student knowledge states over time. They are domain-specific and therefore relate to understanding and reasoning in a particular domain such as algebra, geometry, place value, and rational number (e.g., Clements & Battista, 2000; Clements, Wilson, & Sarama, 2004; Confrey & Maloney, 2010; Daro, Mosher, & Corcoran, 2011; Fuson, 1998; Griffin, 2009; Simon, 1995; Simon & Tzur, 2004). With optimal design, learning trajectories can be used to support formative assessment processes that include connecting observed student performances to domain-referenced (e.g., "student *x* is distance *y* from expected 'expert' performance levels") and individual-referenced (e.g., "student *x* is distance *y* from expected student *x* performance levels given what the teacher understands about the knowledge states of student *x*") ways of acting (Cowie & Bell, 1999). Thus, if a trajectory reveals a diagnostic range of student understanding that a teacher or student is likely to encounter it may provide a basis for instructional responses that promote learning.

Most learning trajectories are designed to directly inform learning and instruction. Indeed, the *goal trajectory* (GT) concept described here is based upon the well-established idea of the learning trajectory, but the GT serves a different purpose which is to make the otherwise implicit models of learning progressions in a math curriculum explicit, an express priority for researchers interested in tracing student knowledge states in the context of a math curriculum. The present paper describes a model for building a GT and explicates its utility for evaluating the variation and growth of mathematical understanding and reasoning in the contexts of particular curriculum-embedded assessments in K–6 math curricula. The research is situated in the context of a larger study designed to address some of the most pressing problems of classroom assessment practice, and is aimed at strengthening the linkages among assessment design, instruction, and student learning.

The current notion of the GT incorporates elements of the developmental progressions that partially compose typical learning trajectory constructs (e.g., Fuson, 1997; Griffin, 2009). Elsewhere, cognition-and-instruction-based design methods have been designed for "forward engineering" a mathematics curriculum (e.g., Clements & Battista, 2000). By contrast our GT serves a purpose of principled retrospective evaluation that is focused on the embedded assessments in an existing mathematic curriculum.

Thus, the current approach to formulating a goal trajectory will be most useful to researchers and practitioners that work in situations where an instructional sequence is present (i.e., in a "scope and sequence") but where a developmental progression—as defined by empirically and theoretically grounded

models of learning—is implicit. The approach consists of six primary design processes: (a) Define the design product, (b) Specify the purpose of the product, (c) Identify the features of the design product, (d) Evaluate, (e) Update, and (f) Classify the features. To illustrate these design processes, we focus on the goals that comprise the algebraic reasoning strand of the standards-driven curriculum, *Everyday Mathematics* (EM; Bell et al., 2007).

Method: Six Design Processes and Model Constraints

Our curriculum-and-instruction-based model for building a goal trajectory has six design processes and two design constraints (see Figure 1). The processes are cognitive activities that are either expressed by an individual or distributed across several people and media.

a. Define Goal Trajectory as the Design Product

The first process, *Define the Design Target*, refers to activities in which the researcher or evaluator articulates what will be designed. In the present case we sought to design a GT that modeled variability and growth in knowledge states in and over time for fifth-graders learning how to reason algebraically in the context of a specific math curriculum. We wanted the GT to be a cognitive model with cognitive units at a level of specificity described by the curriculum. Additionally, we wanted the GT to have properties such that it could be used to estimate variability and growth through its different "levels."

b. Specify the Purpose of the Goal Trajectory

The second process, *Specify the Purpose*, refers to activities in which the researcher or evaluator specifies the aim of the design product. It addresses the question, "Why do we need or desire to design such a product (i.e., the GT)?" In the present case, the purpose of designing a GT that models variability and growth in student knowledge states in and across time was to help us (a) select curriculum-embedded activities, and (b) interpret student performance on the selected assessments. The GT is an important tool in our investigation of the cognitive, instructional, and inferential validity of curriculum-embedded assessments. Thus, in the current situation the purpose was pragmatic. However, in other cases the design product can have empirical, pragmatic, and/or theoretical considerations.

c. Identify the Features of the Goal Trajectory

The third process, *Identify the Features of the Goal Trajectory*, operationalizes the elements of the design product. In the present situation the features were cognitive units and properties of the GT. As mentioned earlier we were concerned with preserving the level of cognitive specificity described by the curriculum. In the context of *Everyday Mathematics* (EM), the cognitive units were tied to the learning goals such as *Use patterns to find basic facts* and *Use rules to complete function tables/machines*. The learning goals comprised the Patterns, Functions, and Algebra (PFA) learning strand in the Grade 5 EM curriculum. Another feature was the ordinal property of the GT. Our intent was to design a GT with ordinal levels that could approximate variation in student performance and growth in cognitive complexity over time.

d. Evaluate Process Outcomes

As shown in Figure 1, the fourth process in the model, *Evaluate*, serves at least two functions. One is to evaluate the agreement between the purpose of the design product (i.e., process b) and its features throughout progress in the design cycle. For example, given the purpose of the design (see section *b. Specify the Purpose*), selecting cognitive units at the larger grain sizes of learning strands (e.g., measurement, number, or geometry) or content threads (e.g., patterns and functions, algebraic notation and solving number sentences, or properties of the arithmetic operations) would not have given the GT the necessary power to model cognitive *variability* in or among students. At those levels the GT would only describe two knowledge states: haves and have-nots. Therefore, it was critical to evaluate each feature of the GT with this constraint in mind.

A second function of the *Evaluate* process is to assess the extent that the design features and the method for assigning them into meaningful levels of the GT is viable given the model's design constraints which are explained below. The dashed circular path indicates that (a) the outcomes of two related

processes are cross-evaluated (e.g., outcomes of processes *c* and *f*) and (b) the decision to move forward with the design depends on the balance of that cross-evaluation; if the balance is positive (i.e., consistent with the scope of the model) then advance, if negative (i.e., inconsistent with the scope of the model) then the model needs to be updated (process *e*).

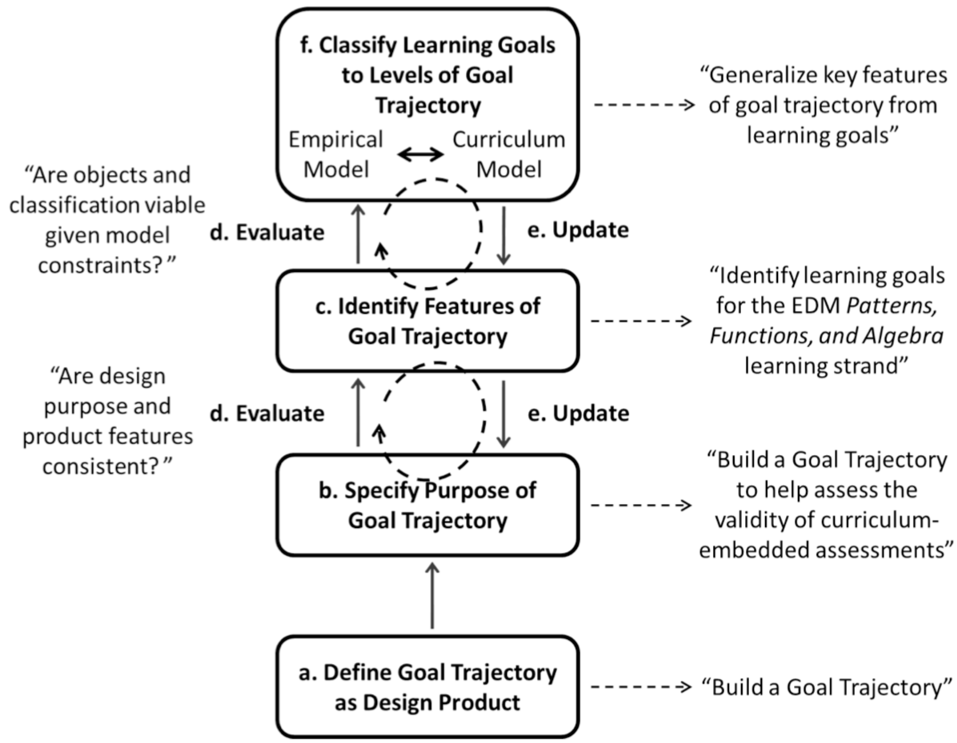

Figure 1. Model of goal trajectory design processes with examples

e. Update

The fifth process, *Update*, serves to make process outcomes consistent with the model or make the model consistent with process outcomes. If an evaluation of two process outcomes reveals an inconsistency (e.g., a learning goal defined as a feature of the GT does not "fit" into a level of the GT), then one or both of those outcomes will need to be updated. In this example a decision may be made to modify a trajectory level, a decision may be made to expand the trajectory by adding a level; or a decision may be made to modify the learning goal. If the two evaluated outcomes are related to processes *c* and *f*, then it may also be necessary to evaluate the outcome of process *b*. This particular chain of evaluations may support a decision to update the purpose of the design (e.g., the GT is useful for selecting embedded assessments but not for interpreting student performance). The cyclic iterations between *Evaluate* and *Update* processes can be one, few, or many in the actual design cycle. Indeed, the model is referred to as a design "cycle" because it is not linear in a strict sense. It is important that researchers or evaluators engaged in the design cycle keep careful records of the model's development from initial conception to final design. In our project we write reports that trace the nature of the design cycle as it unfolds.

Once the learning goals were identified in the curriculum and extracted, we met with the curriculum developers to evaluate (a) the extent that our search for PFA learning goals was exhaustive, (b) our understanding of the curriculum layout, and (c) the degree that the level of learning goal information we decided to use at that point in our design would enable us to build the desired GT. Indeed, our in-depth curriculum analysis revealed several layers of learning goal information. In its early stages, our GT referenced information from all of these layers. However, based on discussions with the curriculum developers we updated the model to include only a single source of learning goal information, the Grade-

Level Goals Chart. Our rationale for this decision was that the Grade-Level Goals Chart highlights the units in which the Grade 5 PFA learning goals are introduced. Using the Grade-Levels Goal chart as our point of reference we were able to “see” the concepts and skills that encompassed the Grade 5 curriculum over time. This satisfied a demand of our model (i.e., build a GT whose levels express ordinal relations) and we were ready to enact the sixth design process.

f. Classify Features into Levels of the Goal Trajectory

The sixth design process in our model for building a goal trajectory is *Classify*. To classify means to abstract a smaller set of cognitive constructs from the learning goals that approximated the major forms of reasoning in the trajectory. The Grade-Level Goals Chart yielded 38 PFA learning goals across the 12 units of the Grade 5 curriculum. The goals were organized into seven general levels of reasoning that were scheduled to be introduced in the PFA trajectory. In effect, the *Classify* process “collapses” all related learning goals across task demand (e.g., recall vs. recognition) and external representation format (e.g., base-10 blocks vs. arrays) resulting in a general set of learning goals and a manageable GT. Notice how Figure 1 indicates that the *Classify* process is constrained by two sources of information: (a) prior research in developmental psychology, cognitive psychology, and mathematics education on the development of and variability in algebraic reasoning (i.e., the “Empirical Model”), and (b) the instructional sequence of key concepts and skills as outlined by the curriculum (i.e., the “Curriculum Model”). As depicted in Figure 1, the resulting learning trajectory was subjected to an Evaluate-Update Cycle before final approval.

Learning Goal Trajectory for Understanding Patterns, Functions and Algebra

The result of the design processes in the current case is the Patterns, Functions, and Algebra (PFA) goal trajectory shown in Table 1. The design processes revealed that the general PFA goal trajectory for acquiring algebraic thinking was implicitly characterized by EM as growth from none or very little understanding of patterns, to identifying and using patterns, to formalizing patterns as a means for solving problems, to generalizing rules from patterns and sequences, to formalizing rules in notational, graphical, and tabular formats, to finally being able to reason with and about variables. The organization of the trajectory was consistent with a growing body of research in cognitive science and mathematics education which suggested that algebra acquisition could be defined by cognitive growth along a multi-path continuum of reasoning with patterns and sequences, generalizing rules from patterns and sequences, representing functions among rules, patterns, and sequences, and formalizing variables to think about functions (Carraher & Schliemann, 1992; Kaput & Blanton, 1999; Moss & McNabb, 2011; Smith & Thompson, 2007; Warren & Cooper, 2008).

Table 1: Goal Trajectory for Understanding Patterns, Functions and Algebra

	Level of Understanding	Examples
6	**Abstract Algebraic Functions** (Represent functions using words, algebraic notation, tables and graphs; represent patterns and rules using algebraic notations; translate from one representation to another; use representations to solve problems involving functions)	• Use a variable to represent unknown quantities to solve problems • Represent an algorithm as a general pattern with variables • Solve linear equations with one unknown and multiple operations using trial-and-error or equivalent equation strategies • Solve problems involving functions using representations; including translating from one representation to another
5	**Algebraic Functions** (Represent functions using words, symbols, tables and graphs; use those representations to solve problems)	• Represent functions using algebraic notations • Use representations of function(s) in tables and graphs to solve problems • Use patterns, tables and graphs to define relationships between volumes of 3D solids or between radius and area; • Represent rates with formulas, tables and graphs
4	**Function Rules** (Describe and/or write rules for functions involving the four basic arithmetic operations; use rules to solve problems)	• Identify and use patterns in graph coordinates to match graphs with situations • Use patterns to identify the relationship between numerators and denominators; use patterns to identify relationships between fractions and decimals • Generate rule for comparing, ordering fractions • Describe the patterns in an area model • Use rules to complete function tables/machines • Use words and symbols to extend patterns/ to describe the operations of Addition, Subtraction, Multiplication and/or Division and/or create/use rules to solve problems
3	**Numeric Pattern Rules** (Use words or symbols to create and/or describe rules for numeric patterns; use rules to extend patterns and solve problems)	• Use words and/or symbols and/or arithmetic notation and extend patterns to describe geometric rules • Use and describe patterns to find sums • Describe number patterns related to exponents and/or use them to solve problems
2	**Numeric Patterns** (Identify, use, expand, describe, or create numeric patterns)	• Complete number sequences • Use patterns to find basic facts • Describe and extend patterns among facts and their extension • Identify and/or use patterns in skip counting • Count in Equal Intervals
1	**No Understanding of Patterns**	• Not able to complete number sequences or count in equal intervals

Relationships Between the PFA Goal Trajectory and the Common Core State Standards

In addition to being consistent with empirical models of growth in algebraic reasoning, the trajectory also aligned with the mathematical content domains and practices outlined by the Common Core State Standards (CCSS) in several interesting ways. First, the Grade 5 EM trajectory for understanding patterns, functions, and algebra embodies two Grade 5 CCSS content domains: *Operations and Algebraic Thinking*

Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Kalamazoo, MI: Western Michigan University.

(OA) which focus on writing and interpreting numerical expressions and analyzing patterns and relationships and *Number and Operations in Base 10*. Second, the Grade 5 goal trajectory relates to these CCSS content domains across Grade 2, Grade 3, and Grade 4 but the mathematical foci (i.e., "clusters") vary among the grades. For example, whereas the CCSS Grade 5 OA domain has two relevant clusters that focus on (a) writing and interpreting numerical expressions, and (b) analyzing patterns and relationships, the CCSS Grade 3 OA domain has four clusters that emphasize (a) representing and solving multiplication and division problems, (b) understanding properties of multiplication and the relationship between multiplication and division, (c) multiplying and dividing using strategies (e.g., $8 \times 4 = 32$ therefore $32 \div 4 = 8$) and properties of operations, and (d) solving for unknown quantities that involve the four operations in addition to identifying and explaining arithmetic patterns. Aspects of the goal trajectory also map onto features of the Grade 6 CCSS content domain, *Expressions and Equations*, which includes clusters that focus on (a) applying and extending what is understood about arithmetic to algebraic expressions, (b) reasoning about and solving one-variable equations and inequalities, and (c) representing and analyzing the relationships between dependent and independent variables.

Besides aligning with the CCSS mathematical *content domains*, we also found the goal trajectory to be well-aligned with the CCSS mathematical *practices*; that is, the various habits of mind that mathematics instructors are expected foster in their students such as constructing viable arguments and reasoning with others, modeling with mathematics, using appropriate tools strategically, and attending to precision. There are various mathematical practices that map onto particular levels of the goal trajectory. For instance, take *Use a variable to represent unknown quantities to solve problems*, taken from the sixth level of understanding in the goal trajectory, *Abstract Algebraic Functions* (Table 1). The level of understanding relates to the CCSS mathematical practice that indicates variables are used to solve problems because they can help *make sense* of quantities and relationships. This mathematical practice implies that variables have greater utility than as simple tools for identifying or recalling answers. A second example of the alignment between the trajectory and the mathematical practices described by the CCSS can be found if one looks at *Complete number sentences* in the *Numeric Patterns* level of understanding in the goal trajectory. The latter is related to the CCSS mathematical practice that promotes the capacity to seek and use structure to describe and extend facts and patterns. The implication is that engaging students in practices that give them opportunities to identify the structure of number sequences should lead to efficient pattern identification strategies that can be applied across different task situations.

Discussion

A six-process model for building curriculum-and-instruction-based goal trajectories for cognitive research and instructional assessment was proposed. We instantiated the processes of the model in the context of our work with the Patterns, Functions, and Algebra learning strand in the Grade 5 *Everyday Mathematics* curriculum. The design processes yielded a unique representation of the goal information that was already represented—albeit, "hidden"—in the organization of the curriculum. Interestingly, the representation that we constructed as the PFA goal trajectory was quite different from the representation of that information as presented by the curriculum.

Re-Presentations of Curriculum-Embedded Goal Structures

Cognitive psychologists have reliably shown that different representations of equivalent information can vary in the way that they preserve information, and this in turn can yield differential affordances for accessing and utilizing the same information (e.g., Larkin & Simon, 1987; Palmer, 1978; Zhang & Norman, 1994). An evaluation of the model proposed in this paper indicates that the benefits of the constructed GT are the result of the aforementioned *representational effect* (Nickerson, 1988; Zhang, 1997). Indeed, the GT affords fresh and important insights into student understanding that expand upon what is available from the *Everyday Mathematics* curriculum materials, while also remaining faithful to the curriculum by basing the GT on the curricular learning goals and instructional materials. For one, the goal trajectory allows us to predict and account for a wider range of student performance on an activity than what is usually estimated by the curriculum, because the curriculum-based representation is typically

limited to dichotomous evaluations of student performance such that student performance either reflects evidence of goal acquisition or it does not. A second benefit of the PFA goal trajectory is that it makes it possible to interpret student performance in terms of the cognitive constructs that are relevant to a particular domain in the contexts of the curriculum and scientific progress. Thus, the goal trajectory affords greater diagnostic information about student performance relative to the learning and acquisition of algebraic thinking.

Investigating Curriculum-CCSS Goal Alignment

Although the CCSS are based on notions of a learning trajectory or progression, their explicit description of one is limited to expectations of mathematical content domains and practices *across* not *within* grades. By comparing our constructed GT to the CCSS it became clear that for a teacher at a particular grade the CCSS was not intended to represent the expected understandings and reasoning patterns of students "well below or well above grade-level expectations," nor was it meant to account for variation contributed by English language learners or children with special needs. We propose that GTs help to illuminate—within the context of a particular mathematics curriculum—the potential for multiple levels of knowledge and reasoning that may be observed as students complete a given activity.

Mapping the CCSS Operations and Algebra content domain onto the GT of an elementary grades math curriculum revealed interesting relationships between each level of the goal trajectory and the CCSS. In particular, as the GT levels progressed, the number of shared relations between each level and the standards increased. Whereas the earlier levels of the trajectory shared a one-to-one relationship with the CCSS standards, the advanced levels of the trajectory shared a one-to-many relationship with the standards in which a single level of the GT was linked to multiple goals in the CCSS. Finally, in support of the CCSS's position about the breadth of mathematical practices, our analysis indicated that the CCSS mathematics practices were differentially instantiated at each GT level of understanding. The extent that these patterns will emerge with other GTs (e.g., Number and Numeration) and the empirical validity of the GT levels is currently being investigated.

Acknowledgments

The authors would like to thank Susan Wortman for helping with some of the earlier curriculum analyses. We are also grateful to Louis V. DiBello, James W. Pellegrino, Susan R. Goldman, and three anonymous reviewers for their thoughtful comments on an earlier draft. This research was supported in part by grants from IES (#R305A090111) and NSF (DRL-0732090). The opinions expressed in this paper are those of the authors and do not necessarily reflect the views of IES or NSF.

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